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CURVATURE INHERITANCE IN FINSLER SPACE

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ABSTRACT

S.B. Misra and A.K. Misra (1984) have studied affine motion in RNP Finsler space. U.P Singh and A. K. Singh (1981) have defined N-curvature collineations and discussed the existence of $N-$curvature collineations of different types in Finsler space. S. P. Singh (2003) has introduced the concept of curvature inheritance in Finsler spaces. J. K. Gatoto and S.P. Singh (2008) have defined and studied curvature inheritance for the relative curvature tensor $\widetilde{K}$ in Finsler space. C. K. Mishra and D. D. Yadav (2007) have investigated and discussed projective curvature inheritance in normal projective Finsler space. In the present paper, the author has defined $N-$curvature inheritance in Finsler space. Some special cases of $N-$curvature inheritance have also been discussed.

KEYWORDS: Finsler Space, Projective Normal Connection Coefficients, Curvature Inheritance, Normal Curvature, Lie-Derivative

1. INTRODUCTION

We consider an dimensional affinely connected Finsler space F_n equipped with $2n$ line-elements $(x,\dot{x})$ of degree one in its directional argument $\dot{x}^i$. The covariant derivative of the tensor field $X^i(x,\dot{x})$ with respect to x^l is given by (Yano, 1957)

$$\nabla_j X^i = \partial_j X^i - \dot{\partial}_m X^i \pi^m_{hj}\dot{x}^h + X^m \pi^i_{mj} \tag{1.1}$$

Where

$$\pi^i_{jk} = G^i_{jk} - \frac{1}{(n+1)}\dot{\partial}_j G^r_{kr}\dot{x}^i \tag{1.2}$$

Is projective normal connection coefficient and satisfy the following relations

$$(a)\dot{\partial}_l \pi^i_{jk}\dot{x}^l = 0,\ (b)\pi^i_{jk} = \pi^i_{kj} \tag{1.3}$$

The symbols $G^i_{jk}(x,\dot{x}) = \dot{\partial}_j\dot{\partial}_k G^i(x,\dot{x})$, given in (1.2) are Berwalds connection parameters (Rund,1959). We shall denote such space by $N-F_n$.

Let us now consider the infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)dt \tag{1.4}$$

Where $v^i(x)$ is a vector field independent of the directional argument $\dot{x}^i$ and dt is an infinitesimal point constant. The Lie-derivative of any tensor field $T^i_j(x,\dot{x})$ and connection parameter π^i_{jk} are given by (Yano, 1957)

$$L_v T^i_j(x,\dot{x}) = \nabla_h T^i_j v^h - T^h_j \nabla_h v^i + T^i_h \nabla_j v^h + (\dot{\partial} T^i_j)\nabla_s v^h \dot{x}^s \tag{1.5}$$

And

$$L_v T^i_j(x,\dot{x}) = \nabla_j \nabla_k v^i - N^i_{hjk} v^h + \pi^i_{sjk} \nabla_p v^s \dot{x}^p \tag{1.6}$$

The commutation formulae involving the Lie-derivative of any tensor $T^i_{jk}(x,\dot{x})$ and connection coefficients $\pi^i_{jk}(x,\dot{x})$ are respectively given by (Yano, 1957)

$$\dot{\partial}_l (L_v T^i_{jk}) - L_v(\dot{\partial}_l T^i_{jk}) = 0 \tag{1.7}$$

$$L_v(\nabla_l T^i_{jk}) - \nabla_l (L_v T^i_{jk}) = T^a_{jk} L_v \pi^i_{al} - T^i_{ak} L_v \pi^a_{jl} \tag{1.8}$$

$$- T^i_{ja} L_v \pi^a_{kl} - (\dot{\partial}_l T^i_{jk})(L_v{}^s \pi^a_{bl})\dot{x}^b$$

And

$$(\nabla_l L_v \pi^i_{kh}) - \nabla_k (L_v \pi^i_{jh}) = L_v N^i_{jkh} - (\dot{\partial}_l \pi^i_{kh})(L_v \pi^l_{bj})\dot{x}^b - (\dot{\partial}_l \pi^i_{jh})(L_v \pi^l_{bk})\dot{x}^b \tag{1.9}$$

Where $N^i_{jkh}(x,\dot{x})$ is normal projective curvature tensor field and satisfies the relation (U.P Singh and A.K. Singh, 1981)

$$(a) N_{kh} = N^i_{ikh},\ (b)\dot{\partial}_r N^i_{jkh}\dot{x}^r = 0 \tag{1.10}$$

And $(c) N^i_{jkh} = -N^i_{kjh}$.

In a Finsler space $N - F_n$, if the curvature tensor field N^i_{jkh} satisfies the relation (Singh and Singh, 1981)

$$\nabla_s N^i_{jkh} = K_s N^i_{jkh}, \tag{1.11}$$

Where K_s is any non-zero vector, then the space under consideration is called recurrent Finsler space and $K_s(x)$ is called recurrence vector. We denote such space by $N - F^*_n$.

2. $N-$ CURVATURE INHERITANCE IN FINSLER SPACE

In this section, we shall define and study an infinitesimal point transformation which is called

$N-$ Curvature Inheritance

Definition 2.1: In a Finsler space $N-F_n$, if the normal curvature tensor field N^i_{jkh} satisfies the relation

$$L_v N^i_{jkh} = \alpha(x) N^i_{jkh}, \tag{2.1}$$

With respect to the vector $v^i(x)$, the infinitesimal point transformation (1.4) is called $N-$curvature inheritance. The entity α(x) is non-zero function.

Definition 2.2: In a Finsler space $N-F_n$ the infinitesimal point transformation (1.4) is said to be an N-Ricci inheritance if there exists a vector field $v^i(x)$ such that

$$L_v N_{kh} = \alpha(x) N_{kh} \tag{2.2}$$

The infinitesimal point transformation (1.4) is said called an affine motion if it satisfies the condition $L_v g_{ij} = 0$. In order that the infinitesimal point transformation (1.4) be an infinitesimal affine motion in a Finslerspace $N-F_n$, it is necessary and sufficient that the Lie-derivative of π^i_{jk} with respect to (1.4) vanishes, that is

$$L_v \pi^i_{jk} = 0$$

In view of the former condition, the equation (1.9) yields

$$L_v N^i_{jkh} = 0 \tag{2.3}$$

Using (2.1) in (2.3), we get

$$\alpha(x) N^i_{jkh} = 0. \tag{2.4}$$

Since $\alpha(x)$ is also non-zero function, it yields $N^i_{jkh} = 0$. Consequently we state

Theorem 2.1: Every affine motion admitted in a Finsler space $N-F_n$, is an $N-$ curvature inheritance if the space is flat.

Applying the commutation formula (1.8) for the normal curvature tensor field N^i_{jkh}, we find

$$L_v(\nabla_s N^i_{jkh}) - \nabla_s(L_v N^i_{jkh}) = N^m_{jkh} L_v \pi^i_{ms} - N^i_{mkh} L_v \pi^m_{js} \tag{2.5}$$

$$- N^i_{jmh} L_v \pi^m_{sl} - (\dot{\partial}_m N^i_{jkh})(L_v \pi^m_{bs}) \dot{x}^b.$$

If the $N-$curvature inheritance is an affine motion, then $L_v \pi^i_{jk} = 0$ is satisfied. In this case equation (2.5) yields

$$L_v(\nabla_s N^i_{jkh}) = \alpha(x)(\nabla_s N^i_{jkh}) \tag{2.6}$$

In view of (2. 1) if the gradient vector $\nabla_s \alpha$ is zero accordingly we have

Lemma 2.1: When an $N-$curvature inheritance admitted in $N-F_n$ becomes an affine motion, the covariant derivative of the normal curvature tensor field N^i_{jkh} satisfies the inheritance relation (2.6) provided the gradient $\nabla_s \alpha$ is zero.

H. Hiramatu (1954) has determined that the necessary and sufficient condition for an infinitesimal point transformation (1.4) to be homothetic transformation is that the relation

$$L_v g_{ij} = 2C g_{ij} \tag{2.7}$$

Where C is some constant holds good. Any solution of (2.7) always satisfies the relation $L_v \pi^i_{jk} = 0$.

Hence from (1.9), we get

$$L_v N^i_{jkh} = 0, \tag{2.8}$$

Which implies $N^i_{jkh} = 0$

Hence we state

Theorem 2.2: Every homothetic transformation admitted in $N-F_n$ is an $N-$ curvature inheritance if the space is flat.

Differentiating (1.11) covariantly with respect to an x^m, we get

$$\nabla_m \nabla_s N^i_{jkh} = (\nabla_m K_s + K_s K_m) N^m_{jkh}$$

Which yields

$$\nabla_{[m} \nabla_{s]} N^i_{jkh} = \nabla_{[m} K_{s]} N^m_{jkh}. \tag{2.9}$$

Taking Lie-derivative of both sides of (2.9), we obtain

$$2\alpha \nabla_{[m} \nabla_{s]} N^i_{jkh} = (L_v \nabla_{[m} K_{s]} + \alpha \nabla_{[m} K_{s]}) N^m_{jkh} \tag{2.10}$$

In view of (2.1) and Theorem 2.2

If we let $L_v \nabla_{[m} K_{s]} = -\alpha \nabla_{[m} K_{s]}$, the above equation reduces to

$$\nabla_{[m} \nabla_{s]} N^i_{jkh} = 0. \tag{2.11}$$

Conversely, if the relation(2.11) is true, then the equation (2.10) assumes the form

$$N^i_{jkh}(L_v \nabla_{[m} K_{s]} + \alpha \nabla_{[m} K_{s]}) = 0 \quad , \tag{2.12}$$

Which implies

$$L_v \nabla_{[m} K_{s]} = -\alpha \nabla_{[m} K_{s]}, \tag{2.13}$$

Since $N - F_n$ is non-flat.

We thus state

Theorem 2.3: In a recurrent Finsler space $N - F_n^*$ which admits $N-$ curvature inheritance, the necessary and sufficient condition for the identity

$$\nabla_{[m} \nabla_{s]} N^i_{jkh} = 0 \tag{2.14}$$

To be true is that the recurrence vector satisfies the inheritance property (2.13).

Applying Lie-operator to (1.11), we get

$$L_v(\nabla_s N^i_{jkh}) = (L_v K_{s]}) N^i_{jkh} + K_s \alpha N^i_{jkh} \tag{2.15}$$

In view of (2.1) and Lemma 2.1, the above equation takes the form

$$\alpha(\nabla_s N^i_{jkh} - K_s N^i_{jkh}) = (L_v K_s) N^i_{jkh}. \tag{2.16}$$

Since the space $N - F_n^*$ is recurrent, the equation (2.16) reduces to

$$(L_v K_s) N^i_{jkh} = 0, \tag{2.17}$$

Which implies

$$N^i_{jkh} = 0 \tag{2.18}$$

In view of the fact $(L_v K_s) \neq 0.$ This contradicts our assumption that the recurrent Finsler space $N - F_n^*$ is non-flat

Consequently we state

Theorem 2.4: A general recurrent Finsler space $N - F_n^*$ does not admit an $N-$ curvature inheritance if it becomes an affine motion.

3. SPECIAL CASES AND DISCUSSIONS

In this section we study and discuss two special cases of curvature inheritance in $N - F_n$ and $N - F_n^*$.

3.1 Contra Field

In a Finsler $N-F_n$, if the vector field $v^i(x)$ satisfies the relation

$$\nabla_j v^i = 0,$$

The vector field $v^i(x)$ determines a contra field.

We now consider a special $N-$curvature inheritance of the form

$$\bar{x}^i = x^i + v^i(x)dt,\ \nabla_j v^i = 0. \tag{3.1}$$

Taking Lie-derivative of the normal projective curvature tensor field N^i_{jkh}, we get

$$L_v N^i_{jkh} = \nabla_l N^i_{jkh} v^l - N^l_{jkh} \nabla_l v^i + N^i_{lkh} \nabla_j v^l + N^i_{jlh} \nabla_k v^l \tag{3.2}$$

$$+ N^i_{jkl} \nabla_h v^l - (\dot{\partial}_l N^i_{jkh}) \nabla_s v^l \dot{x}^s.$$

Using (2.1) and (3.1) in the above equation, we get

$$\alpha N^i_{jkh} = \nabla_l N^i_{jkh} v^l. \tag{3.3}$$

The contraction of indices i and j in (3.3) gives

$$\alpha N_{kh} = \nabla_l N_{kh} v^l. \tag{3.4}$$

On the other hand, from Ricci identity

$$2\nabla_{[j} \nabla_{k]} v^i = N^i_{hjk} v^h, \tag{3.5}$$

We obtain

$$N^i_{hjk} v^h = 0, \tag{3.6}$$

In view of the latter of (3.1)

In (3.5) above, if we set the indices $i = j$ and observe (1.10)(a), we get

$$N_{kh} v^h = 0. \tag{3.7}$$

We accordingly state

Theorem 3.1: In a Finsler space $N-F_n$ which admits $N-$curvature inheritance if the vector field $v^i(x)$ spans a contra field, the following conditions

$(i)(a)\ \alpha N^i_{jkh} = \nabla_l N^i_{jkh} v^l$. $(b)\ \alpha N_{jk} = \nabla_l N_{jk} v^l$.

$(ii)(a)\ N^i_{jkh} v^h. = 0,\ (b)\ N_{hk} v^h = 0,$

Hold good.

In a Finsler space $N - F_n$ which admits $N-$curvature inheritance of the form (3.1), if the $N-$curvature inheritance becomes an affine motion, the condition $L_v N^i_{jkh} = 0$. In such case $N - F_n$ is flat and the equation (3.2) yields

$(a)\ \nabla_l N^i_{jkh} v^l. = 0,\ (b)\ \nabla_l N_{jk} v^l = 0.$

Hence we state

Theorem 3.2: If in a Finsler space $N - F_n$ the $N-$curvature inheritance of the form (3.1) admitted becomes an affine motion, the following conditions

- The Finsler space $N - F_n$ is flat one
- $(a)\ \nabla_l N^i_{jkh} v^l. = 0,\ (b)\ \nabla_l N_{jk} v^l = 0.$

Necessarily holds

Let us consider a Finsler space $N - F^*_n$ which satisfies the recurrence condition (1.11).

Applying condition (1.11) to the equation (3.3) (a) , it yields

$\alpha N^i_{jkh} - K_l v^l N^i_{jkh}. = 0.$

Since the space $N - F^*_n$ is non-flat, the above equation reduces to $\alpha = K_l v^l$.

We thus state

Theorem 3.3: In a non-flat Finsler space $N - F^*_n$ which admits $N-$curvature inheritance, if the vector field $v^i(x)$ spans a contra field , the scalar function $\alpha(x)$ is expressed in the form (3.7).

3.2 Concurrent Field

Under a concurrent field , the vector field satisfies the relation

$$\nabla_j v^i = \lambda \delta^i_j \qquad (3.8)$$

Where λ is a non-zero constant.

In this case we consider an $N-$curvature inheritance of the form

$$\bar{x}^i = x^i + v^i(x)dt,\ \nabla_j v^i = \lambda\delta^i_j. \tag{3.9}$$

If the Finsler space $N-F_n$ admits $N-$curvature inheritance (3.8), then we get

$$\nabla_{[j}\nabla_{k]}v^i = 0 \tag{3.10}$$

And from Ricci identity, we get

$$N^i_{jkh}v^h = 0 \tag{3.11}$$

In a recurrent Finsler space $N-F^*_n$, the covariant derivative of the above equation with respect to x^i yields

$$\lambda N^i_{jkh} = 0 \tag{3.12}$$

In view of (1.11), (3.4) (a) and (3.6)

Taking the Lie-derivative of (3.10) and using (2.1),we obtain

$$\alpha\lambda N^i_{jkh} = 0, \tag{3.13}$$

Which implies that the space $N-F^*_n$ is flat This contradicts the assumption that the space $N-F^*_n$ is non-flat

We thus state

Theorem 3.4: In general a recurrent Finsler space $N-F^*_n$ does not admit an $N-$curvature inheritance of the type (3.9)

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